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Hamiltonian diffeomorphisms of toric manifolds and flag manifolds

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Abstract

If s and n are integers relatively prime and $\mathrm{Ham}(G_s(\mathbb{C}^n))$ is the group of Hamiltonian symplectomorphisms of the Grassmannian manifold $G_s(\mathbb{C}^n)$, we prove that $\sharp\pi_1(\mathrm{Ham}(G_s(\mathbb{C}^n))) \geq n$.

We prove that $\pi_1(\mathrm{Ham}(M))$ contains an infinite cyclic subgroup, when M is the one point blow up of $\mathbb{C}P^3$. We give a sufficient condition for the group $\pi_1(\mathrm{Ham}(M))$ to contain a subgroup isomorphic to $\mathbb{Z}^p$, when M is a general toric manifold.

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1. Introduction

Let (M, ω) be a closed symplectic $2n$-manifold. By $\mathrm{Ham}(M, \omega)$ is denoted the group of Hamiltonian symplectomorphisms of (M, ω) [21,25]. The homotopy type of $\mathrm{Ham}(M, \omega)$ is only completely known in a few particular cases [20,25]. When M is a surface, $\mathrm{Diff}_0(M)$ (the connected component of the identity map in the diffeomorphism group of M) is homotopy equivalent to the symplectomorphism group of M, hence the topology of the groups $\mathrm{Ham}(M)$ in dimension 2 can be deduced from the description of the diffeomorphism groups of surfaces given in [6] (see [25]). On the other hand, positivity of the intersections of J-holomorphic spheres in 4-manifolds played a crucial role in the proof of results about the homotopy type of $\mathrm{Ham}(M)$ when M is a ruled surface (see [9,1,2]). But these arguments which work in dimension 2 or dimension 4 cannot be generalized to higher dimensions.

Using properties of the symplectic action on quantizable manifolds, in [27] we gave a lower bound for $\sharp\pi_1(\mathrm{Ham}(\mathcal{Q}))$, when $\mathcal{Q}$ is a quantizable coadjoint orbit of a semisimple Lie group G and this orbit satisfies some technical hypotheses. By quite different methods McDuff and Tolman have proved the following result: if $\mathcal{O}$ is a coadjoint orbit of the semisimple group G, and the action of G on $\mathcal{O}$ is effective, then the inclusion $G \to \mathrm{Ham}(\mathcal{O})$ induces an injection on π_1 [23]. This result answers a question posed in [29].

Here we give a new approach to determine lower bounds for $\sharp\pi_1(\mathrm{Ham}(\mathcal{O}))$. We will consider curves in G with initial point at e. Every family $\{g_t \mid t \in [0, 1]\}$ of elements of G, with $g_0 = e$ and $g_1 \in Z(G)$ defines a loop ψ in the group $\mathrm{Ham}(\mathcal{O})$. We will use the Maslov index of the linearized flow to deduce conditions under which the loops

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ψ and $\tilde{\psi}$, generated by the families g_t and $\tilde{g}_t$ with different endpoints, are not homotopic. So our lower bounds for $\sharp\pi_1(\mathrm{Ham}(\mathcal{O}))$ will be no bigger than $\sharp Z(G)$.

In particular we consider the group $SU(n+1)$ and an orbit $\mathcal{O}$ diffeomorphic to the Grassmannian $G_s(\mathbb{C}^{n+1})$ of s-dimensional subspaces in $\mathbb{C}^{n+1}$, with s and $n+1$ relatively prime. For each element of $Z(SU(n+1))$ we will construct a curve g_t in $SU(n+1)$ such that the corresponding loops in $\mathrm{Ham}(\mathcal{O})$ relative to different elements of $Z(SU(n+1))$ have distinct Maslov indices. So these loops are homotopically inequivalent, and we have the following theorem.

Theorem 1. *If $\mathcal{O}$ is a coadjoint orbit of $SU(n+1)$ diffeomorphic to the Grassmannian $G_s(\mathbb{C}^{n+1})$ with s and $n+1$ relatively prime, then*

$$\sharp\pi_1(\mathrm{Ham}(\mathcal{O})) \geq n+1. \tag{1.1}$$

In [27] we gave a lower bound for $\sharp\pi_1(\mathrm{Ham}(\mathcal{Q}))$, when $\mathcal{Q}$ is a *quantizable* coadjoint orbit. For the Grassmann manifolds to which the results of [27] are applicable, the bound given in [27] coincides with (1.1). However the results obtained here are more general, since now we do not assume that $\mathcal{O}$ is quantizable. We also give a lower bound for $\sharp\pi_1(\mathrm{Ham}(\mathcal{O}))$ when $\mathcal{O}$ is a coadjoint orbit of $SU(n+1)$ diffeomorphic to a general flag manifold in $\mathbb{C}^{n+1}$.

On the other hand, a loop ψ in the group $\mathrm{Ham}(M,\omega)$ determines a Hamiltonian fibration $E \xrightarrow{\pi} S^2$ with standard fibre M. On the total space E we can consider the first Chern class $c_1(VTE)$ of the vertical tangent bundle of E. Moreover on E is also defined the coupling class $c_\psi \in H^2(E,\mathbb{R})$ [11]. This class is determined by the following properties:

(i) $i_q^*(c_\psi)$ is the cohomology class of the symplectic structure on the fibre $\pi^{-1}(q)$, where i_q is the inclusion of $\pi^{-1}(q)$ in E and q is an arbitrary point of S^2.

(ii) $(c_\psi)^{n+1} = 0$.

These canonical cohomology classes of E determine the characteristic number [19]

$$I_\psi = \int_E c_1(VTE)c_\psi^n. \tag{1.2}$$

I_ψ depends only on the homotopy class of ψ. Moreover I is an $\mathbb{R}$-valued group homomorphism on $\pi_1(\mathrm{Ham}(M,\omega))$, so the non-vanishing of I implies that the group $\pi_1(\mathrm{Ham}(M,\omega))$ is infinite. That is, I can be used to detect the infinitude of the corresponding homotopy group. Furthermore I calibrates the Hofer's norm ν on $\pi_1(\mathrm{Ham}(M,\omega))$ in the sense that $\nu(\psi) \geq C|I_\psi|$, for all ψ, where C is a positive constant [25].

In [28] we gave an explicit expression for the value of the characteristic number I_ψ. This value can be calculated if one has a family of local symplectic trivializations of TM at one's disposal, whose domains cover M and are fixed by the ψ_t's (see Theorem 3 in [28]). In this paper we use this theorem of [28] to prove that $\pi_1(\mathrm{Ham}(M))$ contains an infinite cyclic subgroup, when M is the one point blow up of $\mathbb{C}P^3$. More precisely, in Section 3 we will prove the following result about the Hamiltonian group of the one point blow up of $\mathbb{C}P^3$.

Corollary 2. *Let (M,ω) be the symplectic toric 6-manifold associated to the polytope obtained truncating the tetrahedron of $\mathbb{R}^3$ with vertices $(0,0,0)$, $(\tau,0,0)$, $(0,\tau,0)$, $(0,0,\tau)$ by a horizontal plane [10], then $\pi_1(\mathrm{Ham}(M,\omega))$ contains an infinite cyclic subgroup.*

Using quite different techniques McDuff and Tolman proved this result in [24].

We also give a sufficient condition for $\pi_1(\mathrm{Ham}(M))$ to contain a subgroup isomorphic to $\mathbb{Z}^p$, when M is a general toric manifold. More precisely, let $\mathbb{T}$ be the torus $(S^1)^r$, and $\mathfrak{t} = \mathbb{R}\oplus\cdots\oplus\mathbb{R}$ its Lie algebra. Given $w_j \in \mathbb{Z}^r$, with $j = 1,\ldots,m$ and $\tau \in \mathbb{R}^r$ we put

$$M = \left\{ z \in \mathbb{C}^m : \pi \sum_{j=1}^m |z_j|^2 w_j = \tau \right\} \Big/ \mathbb{T}, \tag{1.3}$$

where the relation defined by $\mathbb{T}$ is

$$(z_j) \simeq (z'_j) \quad \text{iff} \quad \text{there is } \xi \in \mathfrak{t} \text{ such that } z'_j = z_j \mathrm{e}^{2\pi \mathrm{i}\langle w_j,\xi\rangle} \text{ for } j = 1,\ldots,m. \tag{1.4}$$

We will assume that there is an open half space in $\mathbb{R}^r$ which contains all the vectors w_j and that $\{w_j\}_j$ spans $\mathbb{R}^r$. We also assume that τ is a regular value of the map

$$z \in \mathbb{C}^m \mapsto \pi \sum_{j=1}^{m} |z_j|^2 w_j \in \mathbb{R}^r.$$

Then M is a closed toric manifold of dimension $2n := 2(m-r)$ [22].

For $j = 1, \ldots, m$ we put ${}^j\psi_t$ for the symplectomorphism

$${}^j\psi_t : [z] \in M \mapsto [z_1, \ldots, z_j \,\mathrm{e}^{2\pi \mathrm{i} t}, \ldots, z_m] \in M.$$

Let f_j be the corresponding normalized Hamiltonian and

$$\alpha_j := \sum_{k=1}^{m} \left(\int_{\{[z]:z_k=0\}} f_j \omega^{n-1} \right). \tag{1.5}$$

In Section 4 we will prove the following theorem.

Theorem 3. *Let (M, ω) be the toric manifold defined by (1.3) and (1.4). If there are p numbers linearly independent over $\mathbb{Z}$ in the set $\{\alpha_1, \ldots, \alpha_m\}$, where α_j is defined by (1.5), then $\pi_1(\mathrm{Ham}(M, \omega))$ contains a subgroup isomorphic to $\mathbb{Z}^p$. (If $p = 0$ we mean by $\mathbb{Z}^p$ the trivial group.)*

If (M, ω) is the toric manifold determined by the Delzant polytope $\Delta \subset \mathfrak{t}^*$, where T is an n-dimensional torus, we deduce a formula for the value of I on the Hamiltonian loops generated by the effective action of T on M. In this formula geometrical magnitudes relative of Δ and the generator of the loop are involved (Proposition 9).

We denote by $M_H := E\mathrm{Ham}(M) \times_{\mathrm{Ham}(M)} M \to B\mathrm{Ham}(M)$ the universal bundle, with fibre M, over the classifying space $B\mathrm{Ham}(M)$. Let $\mathbf{c} \in H^2(M_H, \mathbb{R})$ denote the coupling class [14]. If G is a compact Lie group and $\phi : G \to \mathrm{Ham}(M)$ be a group homomorphism, then ϕ induces a map $\Phi : BG \to B\mathrm{Ham}(M)$ between the corresponding classifying spaces. By means of this map the class $\mathbf{c}$ induces a class $c_\phi \in H^2(M_\phi)$, where $M_\phi = EG \times_G M$. The class c_ϕ is in fact the coupling class of the Hamiltonian fibration $M_\phi \xrightarrow{p} BG$ [21]. Using c_ϕ and $c_1^\phi(TM)$, the G-equivariant first Chern class of TM, we define

$$R_i(\phi) := p_*((c_1^\phi(TM))^i (c_\phi)^{n-i}),$$

p_* being the integration along the fiber. The classes $R_i(\phi)$ were used in the paper [14] to study the cohomology of classifying spaces, and they are generalizations of the Miller–Morita–Mumford classes. Furthermore $R_i(\phi)$ can be calculated by the localization formula in G-equivariant cohomology.

In the set of all Lie group homomorphisms from G to $\mathrm{Ham}(M)$ we declare that two elements are equivalent if they are homotopic by means of a family of *Lie group homomorphisms* from G to $\mathrm{Ham}(M)$. The quotient set is denoted by $[G, \mathrm{Ham}(M)]_{gh}$. If ϕ and $\tilde{\phi}$ are two Hamiltonian G-actions on M which define the same element in $[G, \mathrm{Ham}(M)]_{gh}$, then $R_i(\phi) = R_i(\tilde{\phi})$. Thus, one has a numerical criterion for two Hamiltonian G-actions not to be equivalent under homotopies consisting of Hamiltonian G-actions. We will prove the existence of pairs of Hamiltonian circle actions in a Hirzebruch surface, which define the same element in $\pi_1(\mathrm{Ham})$ but its classes in $[U(1), \mathrm{Ham}(M)]_{gh}$ are not equal.

If $\mu : M \to \mathfrak{g}^*$ is a moment map for a G-action ϕ on M, we denote with Ω_ϕ the equivariant closed 2-form $\omega + \mu$ [13]. Given $Z \in \mathfrak{g}$ we denote by $\mathcal{R}_\phi(Z)$ the number

$$\mathcal{R}_\phi(Z) := (2\pi \mathrm{i})^{-n} \int_M \mathrm{e}^{\mathrm{i}\Omega_\phi(Z)}. \tag{1.6}$$

When $G = U(1)$ and μ is normalized, Ω_ϕ is a representative of the coupling class c_ϕ. If $X \in \mathfrak{g}$ generates a loop ϕ in $\mathrm{Ham}(\mathcal{O})$, where $\mathcal{O}$ is a regular coadjoint orbit of G, then $\mathcal{R}_\phi(X)$ is the Fourier transform of the orbit and the Harish-Chandra Theorem (see [5]) allows us to calculate $\mathcal{R}_\phi(X)$ in terms of the root structure of $\mathfrak{g}$. Using this fact we give an example of two Hamiltonian circle actions on $\mathbb{C}P^1$ which define the same element in $\pi_1(\mathrm{Ham})$ but they are not homotopic by a family of circle actions.

The paper is organized as follows. In Section 2 we calculate the Maslov index of the linearized flow of certain loops in the Hamiltonian group of flag manifolds, and determine lower bound for the corresponding $\sharp\pi_1(\mathrm{Ham})$.

If M is the one point blow up of $\mathbb{C}P^3$, then M is a submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^3$, and the natural actions of $U(1)$ on the projective spaces define Hamiltonian loops ψ in M. Section 3 is concerned with the determination of I_ψ for these loops. As a consequence of these calculations we deduce that $\mathbb{Z} \subset \pi_1(\mathrm{Ham}(M))$.

In Section 4 we generalize the arguments developed in Section 3 to toric manifolds. From this generalization it follows a sufficient condition, stated in Theorem 3, for the existence of an infinite subgroup in $\pi_1(\mathrm{Ham}(M))$, when M is a toric manifold. Finally we check that this sufficient condition does not hold for $\mathbb{C}P^n$ with $n = 1, 2$. This is consistent with the fact that $\pi_1(\mathrm{Ham}(\mathbb{C}P^n))$ is finite for $n = 1, 2$.

In Section 5 the $R_i(\phi)$ are introduced. If M is a Hirzebruch surface, the toric structure allows us to define two Hamiltonian S^1-actions ϕ, $\tilde{\phi}$, which are not homotopic by means of a set of $U(1)$-actions, since $R_1(\phi) \neq R_1(\tilde{\phi})$. When M satisfies some additional hypotheses we will prove the existence of infinitely many pairs (ζ, ζ') of circle actions on M, such that $[\zeta] = [\zeta'] \in \pi_1(\mathrm{Ham}(M))$, but $[\zeta] \neq [\zeta'] \in [U(1), \mathrm{Ham}(M)]_{gh}$.

Conventions. We use the following conventions. If f_t is a time-dependent Hamiltonian on (M, ω), the corresponding Hamiltonian vector Y_t is defined by

$$\iota_{Y_t}\omega = -\mathrm{d}f_t. \tag{1.7}$$

This time-dependent vector field vector determines the respective family ψ_t of symplectomorphisms by

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = Y_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}. \tag{1.8}$$

If the group G acts on M and $X \in \mathfrak{g}$, by X_M is denoted the vector field whose value at $x \in M$ is

$$X_M(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{-tX}x. \tag{1.9}$$

If the action of G is Hamiltonian, a moment map $\mu : M \to \mathfrak{g}^*$ satisfies $\mathrm{d}\mu(X) = \iota_{X_M}\omega$. According to (1.7) and (1.8) the function $f := \mu(X)$ defines the isotopy ϕ_t^X given by

$$\phi_t^X(x) := \mathrm{e}^{tX}x. \tag{1.10}$$

Given $E \to M$ is a G-equivariant bundle, s is a section of E and $X \in \mathfrak{g}$, we define the action of X on s as the section Xs

$$(Xs)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{e}^{tX}) \cdot s(\mathrm{e}^{-tX}x). \tag{1.11}$$

2. Lower bounds for $\sharp\pi_1(\mathrm{Ham}(\mathcal{O}))$

2.1. Maslov index of the linearized flow

We denote by (M, ω) a closed, connected, symplectic, $2n$-dimensional manifold. Let $\psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Ham}(M, \omega)$ be a loop in the group of Hamiltonian symplectomorphisms at Id. Given $x \in M$, the curve $C := \{\psi_t(x) \mid t \in [0, 1]\}$ is null-homotopic [18]. Let S be a 2-dimensional singular disc in M whose boundary is C, and let $X_1, \ldots, X_{2n}$ be vector fields on S which form a symplectic basis of T_pM for each $p \in S$. Then

$$(\psi_t)_*(X_i(x)) = \sum_k A^k{}_i(t, x)X_k(x_t),$$

with $x_t := \psi_t(x)$ and $A \in Sp(2n, \mathbb{R})$. By ρ will be denoted the usual map $Sp(2n, \mathbb{R}) \to U(1)$ which restricts to the determinant map on $U(n)$ [26]. Setting $a(t, x) := \rho(A(t, x))$, we write $J_\psi(X, x)$ for the winding number of the map $t \in \mathbb{R}/\mathbb{Z} \to a(t, x) \in U(1)$. That is,

$$J_\psi(X, x) = \frac{1}{2\pi\mathrm{i}} \int_0^1 a^{-1}\frac{\partial a}{\partial t}(t, x)\mathrm{d}t.$$

If N is the minimal Chern number of M on spheres, the class of $J_\psi(X, x)$ in $\mathbb{Z}/2N\mathbb{Z}$ only depends on the homotopy class of $[\psi]$. The element in $\mathbb{Z}/2N\mathbb{Z}$ defined by $J_\psi(X, x)$ will be denoted $J[\psi]$ and is the Maslov index of the flow ψ_{t*}.

2.2. Coadjoint orbits

Let G be a compact semisimple Lie group, and η an element of $\mathfrak{g}^*$, the dual of the Lie algebra of G. We denote by $\mathcal{O}$ the coadjoint orbit of η equipped with the standard symplectic structure [17]. This orbit can be identified with G/G_η, where G_η is the stabilizer of η for the coadjoint action of G. The subgroup G_η contains a maximal torus T of G [12]. We have the decomposition of $\mathfrak{g}_\mathbb{C}$ as a direct sum of root spaces

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

with Φ the set of roots determined by T. We denote by $\check{\alpha} \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ the coroot of α. Let $\mathfrak{p}$ be the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\eta(i\check{\alpha})\geq 0} \mathfrak{g}_\alpha.$$

By Z_A is denoted the right invariant vector field on G determined by $A \in \mathfrak{p}$. Since $\mathfrak{p}$ is a subalgebra, $\{Z_A \mid A \in \mathfrak{p}\}$ defines an integrable distribution on G. Its projection onto G/G_η is a complex structure on the orbit $\mathcal{O}$ compatible with the symplectic structure. If P is the parabolic subgroup of $G_\mathbb{C}$ generated by $\mathfrak{p}$, $G_\mathbb{C}/P$ is this complexification of $G/G_\eta = \mathcal{O}$, and

$$T^{1,0}_\eta \simeq \mathfrak{g}_\mathbb{C}/\mathfrak{p} \simeq \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha =: \mathfrak{n},$$

where $\Lambda = \{\beta_1, \ldots, \beta_r\}$ is a subset of Φ. Let $A_1, \ldots, A_r$ be a $\mathbb{C}$-basis for $\mathfrak{n}$, with $A_j \in \mathfrak{g}_{\beta_j}$, then $\{Z_{A_j}\}_j$ is a local frame for $T^{1,0}\mathcal{O}$ on a neighborhood U of η.

If $\{g_t \mid t \in [0, 1]\}$ is a family of elements of G, such that $g_0 = e$ and $g_1 \in Z(G)$, then

$$\{\psi_t : gG_\eta \in G/G_\eta \mapsto g_t g G_\eta \in G/G_\eta\}_{t\in[0,1]}$$

is a loop in $\mathrm{Ham}(\mathcal{O})$. Furthermore

$$(\psi_t)_* Z_{A_j} = Z_{g_t\cdot A_j}, \tag{2.1}$$

with $g \cdot A := \mathrm{Ad}_g A$.

Let $g_t = \exp(C_t)$, with $C_t \in \mathfrak{t}$ and $C_0 = 0$. As $[C_t, E] = \alpha(C_t)E$ if $E \in \mathfrak{g}_\alpha$, then we have

$$\mathrm{Ad}_{g_t} A_j = \exp(\beta_j(C_t))A_j.$$

It follows from (2.1) that for $\nu \in U$ the matrix of $(\psi_t)_*(\nu)$ with respect $\{Z_{A_j}\}_j$ is

$$\mathrm{diag}\left(\mathrm{e}^{\beta_1(C_t)}, \ldots, \mathrm{e}^{\beta_r(C_t)}\right) \in U(r).$$

The Maslov index $J[\psi]$ is the class in $\mathbb{Z}/2N\mathbb{Z}$ of the winding number of the map

$$t \in [0, 1] \mapsto \exp\left(\sum_{j=1}^{r} \beta_j(C_t)\right) = \exp\left(\sum_{\alpha\in\Lambda} \alpha(C_t)\right) \in U(1). \tag{2.2}$$

That is,

$$J[\psi] = \frac{1}{2\pi\mathrm{i}} \sum_{\alpha\in\Lambda} \alpha(C_1) + 2N\mathbb{Z}.$$

We have the following proposition.

Proposition 4. *Let $\mathcal{O}$ be a coadjoint orbit of G whose complexification is $G_{\mathbb{C}}/P$, with $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p} \simeq \bigoplus_{\alpha\in\Lambda}\mathfrak{g}_\alpha$. Let $\{C_t\}_{t\in[0,1]}$ be a curve in $\mathfrak{t}$ such that $C_0 = 0$ and $\exp(C_1) \in Z(G)$. If*

$$\frac{1}{2\pi \mathrm{i}}\sum_{\alpha\in\Lambda}\alpha(C_1) \notin 2N\mathbb{Z},$$

then $g_t = \exp(C_t)$ defines a nontrivial element in $\pi_1(\mathrm{Ham}(\mathcal{O}))$.

2.3. Flag manifolds in $\mathbb{C}^{n+1}$

From now to the end of Section 2 G will be the group $SU(n+1)$, and T the subgroup of diagonal elements. We denote by Δ the usual base of roots; that is, $\Delta = \{\alpha_1, \dots, \alpha_n\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ (we use the notation of [8]). Each subset $I \subset \Delta$ determines a parabolic subgroup P_I of $SL(n+1,\mathbb{C})$. This subgroup is generated by the subalgebra

$$\mathfrak{p}_I = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\tilde{I}}\mathfrak{g}_\alpha,$$

where $\tilde{I}$ consists of all roots that can be written as sums of negative elements in I together with all positive roots [7]. If $I = \Delta - \{\alpha_n\}$, then $\mathfrak{p}_I = \mathfrak{g}l(n,\mathbb{C})$ and $\mathfrak{s}l(n+1,\mathbb{C})/\mathfrak{p}_I$ is isomorphic

$$\bigoplus_{j=1}^{n}\mathfrak{g}_{\beta_j} = \mathfrak{n},$$

with $\beta_j = \epsilon_{n+1} - \epsilon_j$. In this case

$$SL(n+1,\mathbb{C})/P_I \simeq SU(n+1)/U(n) = \mathbb{C}P^n.$$

Next we will determine a lower bound for $\sharp\pi_1(\mathrm{Ham}(\mathcal{O}))$, when $\mathcal{O}$ is a coadjoint orbit of $SU(n+1)$ diffeomorphic to $\mathbb{C}P^n$. Let us take a complex number z such that $z^{n+1} = 1$, and put

$$g_t := \mathrm{diag}\left(z^t, \dots, z^t, z^{-nt}\right) \in T \subset SU(n+1).$$

So $g_1 \in Z(SU(n+1))$, and moreover $g_t = \exp(C_t)$, with

$$C_t = \frac{2k\pi \mathrm{i}t}{n+1}\,\mathrm{diag}(1, \dots, 1, -n), \tag{2.3}$$

where k is any element of $\{0, 1, \dots, n\}$. Then $(\epsilon_{n+1} - \epsilon_j)(C_t) = -2k\pi t\mathrm{i}$, and in this case the map (2.2) is

$$t \in [0,1] \mapsto \exp(-2kn\pi t\mathrm{i}) \in U(1),$$

whose winding number is $-kn$. Hence, for $k = 0, 1, \dots, n$ we obtain loops $\{{}_k\psi_t \mid t \in [0,1]\}$ in $\mathrm{Ham}(\mathcal{O})$ such that the corresponding Maslov indices take the values

$$J[{}_k\psi] = -kn + 2N\mathbb{Z}.$$

The minimal Chern number of $\mathbb{C}P^n$ is equal to $n+1$. As $-kn+2(n+1)\mathbb{Z} \neq -jn+2(n+1)\mathbb{Z}$ for $k \neq j \in \{0, 1, \dots n\}$, then $[{}_k\psi] \neq [{}_j\psi] \in \pi_1(\mathrm{Ham}(\mathcal{O}))$. We have proved the following theorem.

Theorem 5. *If $\mathcal{O}$ is a coadjoint orbit of $SU(n+1)$ diffeomorphic to $\mathbb{C}P^n$, then $\sharp\pi_1(\mathrm{Ham}(\mathcal{O})) \geq n+1$.*

It is known that $\pi_1(\mathrm{Ham}(\mathbb{C}P^1)) = \mathbb{Z}/2\mathbb{Z}$ and that $\mathrm{Ham}(\mathbb{C}P^2)$ has the homotopy type of $PU(3)$ [9], so $\sharp\pi_1(\mathrm{Ham}(\mathbb{C}P^2)) = 3$. The bound given in Theorem 5 is compatible with those facts.

Proof of Theorem 1. Now we consider coadjoint orbits of $SU(n+1)$ which are diffeomorphic to the Grassmannian $G_s(C^{n+1})$ of s-dimensional subspaces of $\mathbb{C}^{n+1}$. Let $\mathfrak{p}$ be the parabolic subalgebra generated by $I = \Delta - \{\alpha_s\}$; that is, we delete the s-node in the Dynkin diagram. (If $s = n$, the corresponding Grassmannian is $\mathbb{C}P^n$.) Now

$$\mathfrak{s}l(n+1,\mathbb{C})/\mathfrak{p} = \bigoplus_{\beta}\mathfrak{g}_\beta$$

with $\beta = \epsilon_j - \epsilon_i$, $j = s+1, \dots, n+1$ and $i = 1, \dots, s$.

With the above notations $(\epsilon_j - \epsilon_i)(C_t) = 0$ for any i, j with $j \neq n+1$. Now the map (2.2) is

$$t \in [0,1] \mapsto \exp(-2kst\pi \mathrm{i}) \in U(1),$$

and its winding number is $-ks$. The minimal Chern number N for the Grassmannian $G_s(\mathbb{C}^{n+1})$ is $n+1$. If s and $n+1$ are relatively prime then

$$\sharp\{-ks + 2(n+1)\mathbb{Z} \mid k = 0, 1, \dots n\} = n+1. \quad \square$$

Given $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{sl}(n+1,\mathbb{C})$ which contains the standard Borel subalgebra, then

$$\mathfrak{sl}(n+1,\mathbb{C})/\mathfrak{p} \simeq \bigoplus_{\beta\in\Lambda} \mathfrak{g}_\beta,$$

where $\Lambda = \Phi \setminus \tilde{I}$.

Given $a \in \{1, \dots, n+1\}$ we put

$$\langle a \rangle = \sharp\{\beta = \epsilon_i - \epsilon_a \in \Lambda\} - \sharp\{\beta = \epsilon_a - \epsilon_j \in \Lambda\}. \tag{2.4}$$

Let $C_t(a)$ be the element of $\mathfrak{t}$ defined by

$$C_t(a) = \frac{2k\pi \mathrm{i}t}{n+1} \operatorname{diag}(1, \dots, 1, -n, 1, \dots, 1), \tag{2.5}$$

where $-n$ is in the position a. The element C_t in (2.3) is equal to $C_t(n+1)$. We consider the curve $g_t = \exp(C_t(a))$, then

$$\sum_{\beta\in\Lambda} \beta(C_t(a)) = 2k\mathrm{i}t\langle a \rangle,$$

and the winding number of the map $t \mapsto \exp \sum \beta(C_t(a))$ is $k\langle a \rangle$. Hence

$$\sharp\pi_1\,(\mathrm{Ham}(SL(n+1,\mathbb{C})/P)) \geq \sharp\{k\langle a \rangle + 2N\mathbb{Z} \mid k = 0, 1, \dots, n\}.$$

So one arrives at the following result.

Theorem 6. *If $\mathcal{O}$ is a coadjoint orbit of $SU(n+1)$ diffeomorphic to the flag manifold $SL(n+1,\mathbb{C})/P$, then*

$$\sharp\pi_1\,(\mathrm{Ham}(\mathcal{O})) \geq \max_{a=1,\dots,n+1} (\sharp\{k\langle a \rangle + 2N\mathbb{Z} \mid k = 0, 1, \dots, n\}),$$

where the integer $\langle a \rangle$ is defined by the parabolic subalgebra $\mathfrak{p}$ by (2.4).

3. Hamiltonian group of the one point blow up of $\mathbb{C}P^3$

Given $\tau, \sigma \in \mathbb{R}_{>0}$, with $\sigma < \tau$, let M be the following manifold

$$M = \{z \in \mathbb{C}^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_5|^2 = \tau/\pi, |z_3|^2 + |z_4|^2 = \sigma/\pi\}/\mathbb{T}, \tag{3.1}$$

where the action of $\mathbb{T} = (S^1)^2$ is defined by

$$(a,b)(z_1, z_2, z_3, z_4, z_5) = (az_1, az_2, abz_3, bz_4, az_5), \tag{3.2}$$

for $a, b \in S^1$.

M is a toric 6-manifold; more precisely, it is the toric manifold associated to the polytope obtained truncating the tetrahedron of $\mathbb{R}^3$ with vertices

$$(0,0,0), (\tau,0,0), (0,\tau,0), (0,0,\tau)$$

by a horizontal plane through the point $(0, 0, \lambda)$, with $\lambda := \tau - \sigma$ [10].

For $0 \neq z_j \in \mathbb{C}$ we put $z_j = \rho_j e^{i\theta_j}$, with $|z_j| = \rho_j$. On the set of points $[z] \in M$ with $z_i \neq 0$ for all i one can consider the coordinates

$$\left(\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2, \frac{\rho_3^2}{2}, \varphi_3\right), \tag{3.3}$$

where the angle coordinates are defined by

$$\varphi_1 = \theta_1 - \theta_5, \qquad \varphi_2 = \theta_2 - \theta_5, \qquad \varphi_3 = \theta_3 - \theta_4 - \theta_5. \tag{3.4}$$

Then the standard symplectic structure on $\mathbb{C}^5$ induces the following form ω on this part of M

$$\omega = \sum_{j=1}^{3} d\left(\frac{\rho_j^2}{2}\right) \wedge d\varphi_j. \tag{3.5}$$

3.1. Darboux coordinates on M

Let $0 < \epsilon << 1$, we write

$$B_0 = \{[z] \in M : |z_j| > \epsilon, \text{ for all } j\}.$$

For each $j \in \{1, 2, 3, 4, 5\}$ we set

$$B_j = \{[z] \in M : |z_j| < 2\epsilon \text{ and } |z_i| > \epsilon, \text{ for all } i \neq j\}.$$

The family $B_0, \dots, B_5$ is not a covering of M, but if $[z] \notin \cup B_k$, then there are i, j, with $i \neq j$ and $|z_i| \leq \epsilon \geq |z_j|$.

We will define Darboux coordinates on $B_0, \dots, B_5$. On B_0 we will consider the well-defined Darboux coordinates (3.3).

On B_1, $\rho_j \neq 0$ for $j \neq 1$; so the angle coordinates φ_2 and φ_3 of (3.4) are well-defined. We define x_1, y_1 by the relation $x_1 + iy_1 := \rho_1 e^{i\varphi_1}$ and $x_1 = 0 = y_1$, if $z_1 = 0$. In this way we take as symplectic coordinates on B_1

$$\left(x_1, y_1, \frac{\rho_2^2}{2}, \varphi_2, \frac{\rho_3^2}{2}, \varphi_3\right).$$

We will also consider the following Darboux coordinates: On B_2

$$\left(\frac{\rho_1^2}{2}, \varphi_1, x_2, y_2, \frac{\rho_3^2}{2}, \varphi_3\right), \quad \text{with } x_2 + iy_2 := \rho_2 e^{i\varphi_2}; \text{ and } x_2 = 0 = y_2, \text{ if } z_2 = 0.$$

On B_3

$$\left(\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2, x_3, y_3\right), \quad \text{where } x_3 + iy_3 := \rho_3 e^{i\varphi_3}.$$

On B_4

$$\left(\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2, x_4, y_4\right), \quad \text{with } x_4 + iy_4 := \rho_4 e^{i\varphi_4} \text{ and } \varphi_4 = \theta_4 - \theta_3 + \theta_5.$$

On B_5

$$\left(x_5, y_5, \frac{\rho_2^2}{2}, \chi_2, \frac{\rho_3^2}{2}, \chi_3\right),$$

where

$$x_5 + iy_5 := \rho_5 e^{i\chi_5}, \qquad \chi_2 = \theta_2 - \theta_1, \qquad \chi_3 = \theta_3 - \theta_1 - \theta_4, \qquad \chi_5 = \theta_5 - \theta_1.$$

If $[z_1, \dots, z_5]$ is a point of

$$M \setminus \bigcup_{i=0}^{5} B_i,$$

then there are $a \neq b \in \{1, \dots, 5\}$ such that $|z_a|, |z_b| \leq \epsilon$. We can cover the set $M \setminus \bigcup B_i$ by Darboux charts denoted $B_6, \dots, B_q$ similar to the preceding B_i's satisfying the following condition. The image of each B_a, with $a = 6, \dots, q$, is contained in a prism of $\mathbb{R}^6$ of the form

$$\prod_{i=1}^{6} [c_i, d_i],$$

where at least two intervals $[c_i, d_i]$ have length of order ϵ.

By the infinitesimal "size" of the B_j, for $j \geq 1$, it turns out

$$\int_{B_j} \omega^3 = O(\epsilon), \quad \text{for } j \geq 1. \tag{3.6}$$

3.2. *A loop in* Ham(M)

Let ψ_t be the symplectomorphism of M defined by

$$\psi_t[z] = [z_1 e^{2\pi i t}, z_2, z_3, z_4, z_5]. \tag{3.7}$$

Then $\{\psi_t\}_t$ is a loop in the group $\mathrm{Ham}(M)$ of Hamiltonian symplectomorphisms of M. By f is denoted the corresponding normalized Hamiltonian function. Hence $f = \pi\rho_1^2 - \kappa$ with $\kappa \in \mathbb{R}$ such that $\int_M f\omega^3 = 0$.

We will calculate I_ψ using the following result proved in [28] (Theorem 3 of [28]).

Theorem 7. *Let* $\psi : S^1 \to \mathrm{Ham}(M, \omega)$ *be a closed Hamiltonian isotopy generated by the normalized time-dependent Hamiltonian* f_t. *If* $\{B_1, \dots, B_m\}$ *is a set of symplectic trivializations for* TM *which covers* M *and such that* $\psi_t(B_j) = B_j$, *for all* t *and all* j, *then*

$$I_\psi = \sum_{i=1}^{m} J_i \int_{B_i \setminus \cup_{j<i} B_j} \omega^n + \sum_{i<k} N_{ik}, \tag{3.8}$$

where

$$N_{ik} = n \frac{\mathrm{i}}{2\pi} \int_{S^1} \mathrm{d}t \int_{A_{ik}} (f_t \circ \psi_t)(\mathrm{d}\log r_{ik}) \wedge \omega^{n-1},$$

$A_{ik} = (\partial B_i \setminus \cup_{r<k} B_r) \cap B_k$, *$J_i$ is the Maslov index of $(\psi_t)_*$ in the trivialization B_i and r_{ik} the corresponding transition function of* $\det(TM)$.

We will prove that, in the case we are considering, some summands in (3.8) are of order ϵ. We will neglect the order ϵ summands, and in this way we will obtain an expression which is equal to I_ψ up to an addend of order ϵ.

In the coordinates (3.3) of B_0, ψ_t is the map $\varphi_1 \mapsto \varphi_1 + 2\pi t$. So the Maslov index $J_{B_0} = 0$. It follows from (3.6) and Theorem 7

$$I_\psi = \sum_{i<k} N_{ik} + O(\epsilon), \tag{3.9}$$

with

$$N_{ik} = \frac{3\mathrm{i}}{2\pi} \int_{A_{ik}} f \,\mathrm{d}\log r_{ik} \wedge \omega^2.$$

If $[z] \in A_{ik} \subset \partial B_i \cap B_k$, with $1 \le i < k$, then at least the modules $|z_a|$ and $|z_b|$ of two components of $[z]$ are of order ϵ; so N_{ik} is of order ϵ when $1 \le i < k$. Analogously N_{0k} is of order ϵ, for $k = 6, \ldots, q$. Hence (3.9) reduces to

$$I_\psi = \sum_{k=1}^{5} N_{0k} + O(\epsilon). \tag{3.10}$$

If we put

$$N'_{0k} = \frac{3\mathrm{i}}{2\pi} \int_{A'_{0k}} f \,\mathrm{d}\log r_{ik} \wedge \omega^2, \tag{3.11}$$

with

$$A'_{0k} = \{[z] \in M : |z_k| = \epsilon, |z_r| > \epsilon \text{ for all } r \neq k\},$$

then

$$N_{0k} = N'_{0k} + O(\epsilon)$$

and

$$I_\psi = \sum_{k=1}^{5} N'_{0k} + O(\epsilon). \tag{3.12}$$

3.3. Calculation of the N'_{0k}'s

First we determine the value of N'_{01}. To know the transition function r_{01} one needs the Jacobian matrix R of the transformation

$$\left(x_1, y_1, \frac{\rho_2^2}{2}, \varphi_2, \frac{\rho_3^2}{2}, \varphi_3\right) \to \left(\frac{\rho_1^2}{2}, \varphi_1, \frac{\rho_2^2}{2}, \varphi_2, \frac{\rho_3^2}{2}, \varphi_3\right)$$

in the points of A'_{01}; where $\rho_1^2 = x_1^2 + y_1^2$, $\varphi_1 = \tan^{-1}(y_1/x_1)$. The function $r_{01} = \rho(R)$, where $\rho : Sp(6, \mathbb{R}) \to U(1)$ is the map which restricts to the determinant on $U(3)$ [26]. The non-trivial block of R is the diagonal one

$$\begin{pmatrix} x_1 & y_1 \\ r & s \end{pmatrix},$$

with $r = -y_1(x_1^2 + y_1^2)^{-1}$ and $s = x_1(x_1^2 + y_1^2)^{-1}$. The non-real eigenvalues of R are

$$\lambda_\pm = \frac{x_1 + s}{2} \pm \frac{\mathrm{i}\sqrt{4 - (s + x_1)^2}}{2}.$$

These non-real eigenvalues occur when $(s + x_1)^2 < 2$. On A'_{01} this condition is equivalent to $|\cos\varphi_1| < 2\epsilon(\epsilon^2 + 1)^{-1} =: \delta$, since $\rho_1 = \epsilon$ for the points of A'_{01}.

If $y_1 > 0$ then λ_- of the first kind (see [26]) and λ_+ is of the first kind if $y_1 < 0$. Hence, on A'_{01},

$$\rho(R) = \begin{cases} \lambda_+|\lambda_+|^{-1} = x + \mathrm{i}y, & \text{if } |\cos\varphi_1| < \delta \text{ and } y_1 < 0; \\ \lambda_-|\lambda_-|^{-1} = x - \mathrm{i}y, & \text{if } |\cos\varphi_1| < \delta \text{ and } y_1 > 0; \\ \pm 1, & \text{otherwise;} \end{cases}$$

where $x = \delta^{-1}\cos\varphi_1$, and $y = \sqrt{1 - x^2}$.

If we put $\rho(R) = \mathrm{e}^{\mathrm{i}\gamma}$ then, for the points of A'_{01} in which $|\cos\varphi_1| < \delta$,

$$\cos\gamma = \delta^{-1}\cos\varphi_1, \quad \text{and} \quad \sin\gamma = \begin{cases} -\sqrt{1 - \cos^2\gamma}, & \text{if } \sin\varphi_1 > 0; \\ \sqrt{1 - \cos^2\gamma}, & \text{if } \sin\varphi_1 < 0. \end{cases}$$

So, when φ_1 runs anticlockwise from 0 to 2π, γ goes round the circumference clockwise; that is, $\gamma = h(\varphi_1)$, where h is a function such that

$$h(0) = 2\pi, \quad \text{and} \quad h(2\pi) = 0. \tag{3.13}$$

As $r_{01} = \rho(R)$, then $\mathrm{d}\log r_{01} = \mathrm{i}\mathrm{d}h$.

On A'_{01} the symplectic form (3.5) reduces to $(1/2)(\mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2 + \mathrm{d}\rho_3^2 \wedge \mathrm{d}\varphi_3)$. From (3.11) one deduces

$$N'_{01} = \frac{3\mathrm{i}}{4\pi} \int_{A'_{01}} \mathrm{i} f \frac{\partial h}{\partial \varphi_1} \mathrm{d}\varphi_1 \wedge \mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2 \wedge \mathrm{d}\rho_3^2 \wedge \mathrm{d}\varphi_3. \tag{3.14}$$

The submanifold A'_{01} is oriented as a subset of ∂B_0 and the orientation of B_0 is the one defined by ω^3, that is, by

$$\mathrm{d}\rho_1^2 \wedge \mathrm{d}\varphi_1 \wedge \mathrm{d}\rho_2^2 \wedge \mathrm{d}\varphi_2 \wedge \mathrm{d}\rho_3^2 \wedge \mathrm{d}\varphi_3.$$

Since $\rho_1 > \epsilon$ for the points of B_0, then A'_{01} is oriented by $-\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2^2 \wedge \mathrm{d}\varphi_2 \wedge \mathrm{d}\rho_3^2 \wedge \mathrm{d}\varphi_3$. On the other hand, the Hamiltonian function $f = -\kappa + O(\epsilon)$ on A'_{01}. Then it follows from (3.14) together with (3.13)

$$N'_{01} = 6\pi^2 \kappa \int_0^{\sigma/\pi} \mathrm{d}\rho_3^2 \int_0^{\tau/\pi - \rho_3^2} \mathrm{d}\rho_2^2 + O(\epsilon)$$

that is,

$$N'_{01} = 3\kappa(\tau^2 - \lambda^2) + O(\epsilon). \tag{3.15}$$

The contributions $N'_{02}, N'_{03}, N'_{04}, N'_{05}$ to (3.12) can be calculated in a similar way. One obtains the following results up to addends of order ϵ

$$N'_{02} = N'_{05} = -(\tau^3 - \lambda^3) + 3\kappa(\tau^2 - \lambda^2), \qquad N'_{03} = \tau^2(3\kappa - \tau), \qquad N'_{04} = \lambda^2(3\kappa - \lambda). \tag{3.16}$$

As I_ψ is independent of ϵ, it follows from (3.12), (3.15) and (3.16)

$$I_\psi = 6\kappa(2\tau^2 - \lambda^2) + \lambda^3 - 3\tau^3. \tag{3.17}$$

On the other hand, straightforward calculations give

$$\int_M \omega^3 = (\tau^3 - \lambda^3), \quad \text{and} \quad \int_M \pi\rho_1^2 \omega^3 = \frac{1}{4}(\tau^4 - \lambda^4).$$

So

$$\kappa = \frac{1}{4}\left(\frac{\tau^4 - \lambda^4}{\tau^3 - \lambda^3}\right). \tag{3.18}$$

It follows from (3.17) and (3.18)

$$I_\psi = \frac{\lambda^2(-3\tau^4 + 8\tau^3\lambda - 6\tau^2\lambda^2 + \lambda^4)}{2(\tau^3 - \lambda^3)}. \tag{3.19}$$

Hence I_ψ is a rational function of τ and λ. It is easy to check that its numerator does not vanish for $0 < \lambda < \tau$. So we have proved the following proposition.

Proposition 8. *If ψ is the closed Hamiltonian isotopy defined in (3.7), then the characteristic number $I_\psi \neq 0$.*

Proof of Corollary 2. By Proposition 8 $I_\psi \neq 0$. As I is a group homomorphism on $\pi_1(\mathrm{Ham}(M, \omega))$, then the class $[\psi^l] \in \pi_1(\mathrm{Ham}(M, \omega))$ does not vanish, for all $l \in \mathbb{Z} \setminus \{0\}$. □

4. Hamiltonian group of toric manifolds

In this section we generalize the calculations carried out in Section 3 for the 6-manifold one point blow up of $\mathbb{C}P^3$ to a general toric manifold. Now (M, ω) will denote the toric manifold defined by (1.3) and (1.4).

When $0 \neq z_b \in \mathbb{C}$, we write $z_b = \rho_b e^{i\theta_b}$. The standard symplectic form on $\mathbb{C}^m$ gives rise to the symplectic structure ω on M. On

$$\{[z] \in M : z_j \neq 0 \text{ for all } j\}$$

ω can be written as in (3.5)

$$\omega = \sum_{i=1}^{n} d\left(\frac{\rho_{ai}^2}{2}\right) \wedge d\varphi_{ai},$$

with φ_{ai} a linear combination of the θ_c's.

Given $0 < \epsilon << 1$, we set

$$B_0 = \{[z] \in M : |z_j| > \epsilon \text{ for all } j\}$$
$$B_k = \{[z] \in M : |z_k| < 2\epsilon, |z_j| > \epsilon \text{ for all} j \neq k\},$$

as in Section 3. On B_0 we will consider the Darboux coordinates

$$\left\{\frac{\rho_{ai}^2}{2}, \varphi_{ai}\right\}_{i=1,\ldots,n}.$$

Given $k \in \{1, \ldots, m\}$ we write ω in the form

$$\omega = d\left(\frac{\rho_k^2}{2}\right) \wedge d\varphi_k + \sum_{i=1}^{n-1} d\left(\frac{\rho_{ki}^2}{2}\right) \wedge d\varphi_{ki},$$

where φ_k and φ_{ki} are linear combinations of the θ_c's. Then we consider on B_k the following Darboux coordinates

$$\left\{x_k, y_k, \frac{\rho_{ki}^2}{2}, \varphi_{ki}\right\}_{i=1,\ldots,n-1},$$

with x_k, y_k defined by $x_k + iy_k := \rho_k e^{i\varphi_k}$, if $z_k \neq 0$ and $x_k = 0 = y_k$, if $z_k = 0$.

We denote by ψ_t the map

$$\psi_t : [z] \in M \mapsto [z_1 e^{2\pi i t}, z_2, \ldots, z_m] \in M.$$

$\{\psi_t : t \in [0, 1]\}$ is a loop in $\mathrm{Ham}(M)$. By repeating the arguments of Section 3 one obtains

$$I_\psi = \sum_{k=1}^{m} N'_{0k} + O(\epsilon),$$

where

$$N'_{0k} = \frac{ni}{2\pi} \int_{A'_{0k}} f \, d \log r_{0k} \wedge \omega^{n-1},$$
$$A'_{0k} = \{[z] \in M : |z_k| = \epsilon, |z_j| > \epsilon \text{ for all } j \neq k\},$$

and $f = \pi\rho_1^2 - \kappa_1$, with

$$\int_M \pi\rho_1^2 \omega^n = \kappa_1 \int_M \omega^n.$$

As in Section 3, on A'_{0k} the exterior derivative $\mathrm{d}\log r_{0k} = \mathrm{i}h'(\varphi_k)\mathrm{d}\varphi_k$, where $h = h(\varphi_k)$ is a function such that $h(0) = 2\pi$, $h(2\pi) = 0$. Then

$$N'_{0k} = -n\int_{\{[z]:z_k=0\}} f\omega^{n-1} + O(\epsilon),$$

where $\{[z] \in M : z_k = 0\}$ is oriented by the restriction of ω to this submanifold. Since I_ψ is independent of ϵ, we obtain

$$I_\psi = -n\sum_{k=1}^{m}\left(\int_{\{[z]:z_k=0\}} (\pi\rho_1^2 - \kappa_1)\omega^{n-1}\right). \tag{4.1}$$

For $j = 1, \ldots, m$ we write

$$\alpha_j := \sum_{k=1}^{m}\left(\int_{\{[z]:z_k=0\}} (\pi\rho_j^2 - \kappa_j)\omega^{n-1}\right), \tag{4.2}$$

where κ_j is defined by the condition

$$\int_M \pi\rho_j^2\omega^n = \kappa_j\int_M \omega^n.$$

Proof of Theorem 3. Let us assume that $\alpha_1, \ldots, \alpha_p$ are linearly independent over $\mathbb{Z}$. For $j = 1, \ldots, p$ we put

$${}^j\psi_t : [z] \in M \mapsto [z_1, \ldots, z_j e^{2\pi it}, \ldots, z_m] \in M.$$

Given $q = (q_1, \ldots, q_p) \in \mathbb{Z}^p$ we denote by ψ^q the path product

$$({}^1\psi)^{q_1} \star \cdots \star ({}^p\psi)^{q_p}.$$

Formula (4.1) together with the fact that I is a group homomorphism give

$$I_{\psi^q} = -n\sum_{i=1}^{p} q_i\alpha_i.$$

Analogously if $q' = (q'_1, \ldots, q'_p) \in \mathbb{Z}^p$, then $I_{\psi^{q'}} = -n\sum_{i=1}^{p} q'_i\alpha_i$. By the linear independence of $\alpha_1, \ldots, \alpha_p$ from $I_{\psi^{q'}} = I_{\psi^q}$ it follows $q = q'$. So ψ^q is homotopic to $\psi^{q'}$ iff $q = q'$. □

Example. We will check the above result calculating the family $\{\alpha_j\}$ defined in (4.2) in two particular cases: when the manifold is $\mathbb{C}P^1$ and when it is $\mathbb{C}P^2$.

For

$$M = \mathbb{C}P^1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = \tau/\pi\}/S^1,$$

we have

$$\int_M \pi\rho_1^2\omega = \tau^2/2, \qquad \int_M \omega = \tau.$$

Thus $\kappa_1 = \tau/2$ and $\alpha_1 = -\kappa_1 + \tau - \kappa_1 = 0$. Similarly $\alpha_2 = 0$. In this case the number p in Theorem 3 is 0. This is compatible with the fact that $\pi_1(\mathrm{Ham}(\mathbb{C}P^1)) = \mathbb{Z}/2\mathbb{Z}$.

For

$$M = \mathbb{C}P^2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = \tau/\pi\}/S^1,$$

we have the following values for the integrals involved in the definition of α_1

$$\int_M \omega^2 = \tau^2, \qquad \int_M \pi\rho_1^2\omega^2 = \tau^3/3.$$

So $\kappa_1 = \tau/3$. Moreover for $k \in \{1, 2, 3\}$

$$\int_{\{[z]:z_k=0\}} \omega = \tau.$$

On the other hand, for $k = 2, 3$

$$\int_{\{[z]:z_k=0\}} \pi\rho_1^2\omega = \tau^2/2.$$

So $\alpha_1 = -\kappa_1\tau + (\tau^2/2 - \kappa_1\tau) + (\tau^2/2 - \kappa_1\tau) = 0$. Analogously $\alpha_2 = \alpha_3 = 0$, so p in Theorem 3 is also 0. This result is consistent with the finiteness of $\pi_1(\mathrm{Ham}(\mathbb{C}P^2))$, for $\mathrm{Ham}(\mathbb{C}P^2)$ has the homotopy type of $PU(3)$ [9].

Remark. On the manifold M one point blow up of $\mathbb{C}P^3$, defined by (3.1) and (3.2), one can consider the loop $\tilde{\psi}$ defined by

$$\tilde{\psi}_t[z] = [z_1, z_2, z_3 \mathrm{e}^{2\pi \mathrm{i}t}, z_4, z_5]. \tag{4.3}$$

A similar calculation to the one carried out in the proof of (3.19) shows that $I_{\tilde{\psi}} = -3I_\psi$.

In the definition of M the variables z_1, z_2, z_5 play the same role. However we can consider the following S^1-action on M

$$\hat{\psi}_t[z] = [z_1, z_2, z_3, z_4 \mathrm{e}^{2\pi \mathrm{i}t}, z_5], \tag{4.4}$$

and it turns out that $I_{\hat{\psi}} = 3I_\psi$. Thus Theorem 3 guaranties that only $\mathbb{Z}$ is contained in $\pi_1(\mathrm{Ham}(M))$.

Let (M, ω) be the toric manifold determined by the Delzant polytope $\Delta \subset \mathfrak{t}^*$, where T is an n-dimensional torus. Next we give a formula for the value of I on the Hamiltonian loops generated by the effective action of T on M, in which are involved geometrical magnitudes relative to Δ and the generator of the loop.

By $\mu : M \to \mathfrak{t}^*$ is denoted the moment map for the T-action. Let $\mathbf{b}$ be an element of the integer lattice of $\mathfrak{t}$, and let $\psi_{\mathbf{b}}$ the S^1-action determined by $\mathbf{b}$. The corresponding normalized Hamiltonian function is $f = \langle \mu, \mathbf{b}\rangle - \kappa$, with

$$\int_M \langle \mu, \mathbf{b}\rangle \omega^n = \kappa \int_M \omega^n.$$

Since

$$\int_M \mu\omega^n = \mathrm{C\,m}(\Delta) \int_M \omega^n,$$

where $\mathrm{C\,m}(\Delta)$ is the center of mass of Δ, it follows $\kappa = \langle \mathrm{C\,m}(\Delta), \mathbf{b}\rangle$.

According to (4.1)

$$I_{\psi_{\mathbf{b}}} = -n \sum_{k=1}^{m} \int_{D_k} (\langle \mu, \mathbf{b}\rangle - \langle \mathrm{C\,m}(\Delta), \mathbf{b}\rangle)\, \omega^{n-1},$$

where $D_k := \mu^{-1}(F_k)$, and $F_1, \ldots, F_m$ are the facets of Δ.

We define $\mathrm{C\,m}(D_k)$ by the relation

$$\mathrm{C\,m}(D_k) \int_{D_k} \omega^{n-1} = \int_{D_k} \mu\omega^{n-1},$$

and

$$\mathrm{Vol}(D_k) := \frac{1}{(n-1)!} \frac{1}{(2\pi)^{n-1}} \int_{D_k} \omega^{n-1}.$$

Then

$$I_{\psi_{\mathbf{b}}} = n!(2\pi)^{n-1} \sum_{k=1}^{m} \langle \mathrm{C\,m}(\Delta) - \mathrm{C\,m}(D_k), \mathbf{b}\rangle \mathrm{Vol}(D_k). \tag{4.5}$$

Thus we have the following proposition.

Proposition 9. *Let (M, ω) be the toric manifold associated to the polytope Δ. If there is* **b** *in the integer lattice of* $\mathfrak{t}$ *such that*

$$\sum_{k=1}^{m} \langle \mathrm{C\,m}(\Delta) - \mathrm{C\,m}(D_k), \mathbf{b} \rangle \mathrm{Vol}(D_k) \neq 0,$$

then **b** *generates an element of infinite order in the group* $\pi_1(\mathrm{Ham}(M, \omega))$.

5. Hamiltonian G-actions

Let G be a compact Lie group and $\phi : G \to \mathrm{Ham}(M, \omega)$ a Hamiltonian G-action on M. The group homomorphism ϕ induces a map

$$\Phi : BG \to B\mathrm{Ham}(M, \omega)$$

between the corresponding classifying spaces.

On the other hand, one has the universal bundle with fibre M

$$\begin{array}{ccc} M & \longrightarrow & M_H := E\mathrm{Ham}\,(M) \times_{\mathrm{Ham}\,(M)} M \\ & & \downarrow \pi_H \\ & & B\mathrm{Ham}(M), \end{array}$$

where $E\mathrm{Ham}(M) \to B\mathrm{Ham}(M)$ is the universal principal bundle of the group $H := \mathrm{Ham}(M, \omega)$.

The pullback $\Phi^{-1}(M_H)$ of M_H by Φ is a bundle on BG which can be identified with $p : M_\phi := EG \times_G M \to BG$. Thus we have the following commutative diagram

$$\begin{array}{ccc} M_\phi & \xrightarrow{\Phi'} & M_H \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_H} \\ BG & \xrightarrow[\Phi]{} & BH. \end{array}$$

There exists a unique class $\mathbf{c} \in H^2(M_H, \mathbb{R})$ [14] called the coupling, such that $\mathbf{c}$ extends the fiberwise class $[\omega]$ and $\pi_{H_*}\mathbf{c}^{n+1} = 0$ (where π_{H_*} is the fiber integration). We put c_ϕ for the pullback of $\mathbf{c}$ by Φ'; that is, $c_\phi = \Phi'^*(\mathbf{c}) \in H^2(M_\phi, \mathbb{R})$. Since

$$p_*(c_\phi^{n+1}) = \Phi^*(\pi_{H_*}\mathbf{c}^{n+1}) = 0,$$

c_ϕ is the coupling class of the Hamiltonian fibration $M_\phi \to BG$ [21].

We can also consider the vector bundle

$$(TM)_\phi := EG \times_G TM \to M_\phi.$$

The first Chern class $c_1((TM)_\phi)$ is the G-equivariant first Chern class of TM, and it will be denoted by c_1^ϕ.

By $\mathrm{Hom}(G, \mathrm{Ham}(M))$ is denoted the set of all Lie group homomorphisms ϕ from G to $\mathrm{Ham}(M, \omega)$. In $\mathrm{Hom}(G, \mathrm{Ham}(M))$ one defines the following equivalence relation:

$\phi \simeq \tilde{\phi}$ iff there is a continuous family $\{\phi^s : G \to \mathrm{Ham}(M)\}_{s \in [0,1]}$ of Lie group homomorphisms, such that $\phi^0 = \phi$ and $\phi^1 = \tilde{\phi}$; that is, iff ϕ and $\tilde{\phi}$ are homotopic by a family of *group homomorphisms*. We denote by $[G, \mathrm{Ham}(M)]_{gh}$ the corresponding quotient set. This space is just a set of connected components of the space of homomorphisms from G to $\mathrm{Ham}(M, \omega)$.

If $\phi \simeq \tilde{\phi}$, then the bundles $\tilde{\Phi}^{-1}(M_H)$ and $\Phi^{-1}(M_H)$ are isomorphic. Moreover the isomorphism $M_\phi \to M_{\tilde{\phi}}$ applies $c_{\tilde{\phi}}$ in c_ϕ and $c_1^{\tilde{\phi}}$ in c_1^ϕ.

For $j = 0, 1, \dots, n$ we put

$$\beta_j(\phi) := \left(c_1^\phi\right)^j \left(c_\phi\right)^{n-j} \in H^{2n}(M_\phi, \mathbb{R}).$$

We write $R_j(\phi) := p_*(\beta_j(\phi)) \in H^0(BG)$. By the localization formula in G-equivariant cohomology [5,13]

$$R_i(\phi) = \sum_Z p_*^Z \left(\frac{\beta|_Z}{e_Z} \right), \tag{5.1}$$

where Z varies in the set of connected components of the fixed point set, $p_*^Z : H_G(Z) \to H(BG)$ is the fiber integration on Z, and e_Z is the equivariant Euler class of the normal bundle to Z in M.

From the preceding arguments it follows the following theorem.

Theorem 10. *Given ϕ and $\tilde{\phi}$ two Hamiltonian G-actions on M, if there are $j \in \{0, 1, \ldots, n\}$ and $X \in \mathfrak{g}$ such that $R_j(\phi)(X) \neq R_j(\tilde{\phi})(X)$, then $[\phi] \neq [\tilde{\phi}] \in [G, \mathrm{Ham}(M)]_{gh}$.*

If $\hat{\omega} \in H^2(M_\phi, \mathbb{R})$ is an element which restricts to the class of the symplectic form on the fiber in the fibration $p : M_\phi \to BG$, then

$$c_\phi = \hat{\omega} - \frac{1}{k} p^*(p_*(\hat{\omega}^{n+1})),$$

where the constant $k = (n+1) \int_M \omega^n$ (see [14]). In particular, if $G = U(1)$ we denote by f the normalized Hamiltonian function; that is, $\iota_Y \omega = -\mathrm{d}f$ and $\int_M f\omega^n = 0$, where Y the vector field on M generated by ϕ. Then c_ϕ is the class in $H^2(M_\phi)$ defined by the $U(1)$-equivariant 2-form $\omega + fu$, where u is a coordinate on the Lie algebra $\mathfrak{u}(1)$ dual of a fixed base X of $\mathfrak{u}(1)$ (see [16,13]).

When $G = U(1)$ a representative of $c_1^\phi(\det(TM))$ can be constructed following [3] or [5]. Let s be a local section of $\det(TM)$ over the open V. The infinitesimal action of X on the section s is the section Xs defined in (1.11). Xs is a section which can be written as the product $L \cdot s$, of a function L on V and s. If α is the form relative to s of an equivariant connection on $\det(TM)$, and X_M is the Hamiltonian vector field on M defined in (1.9), then

$$\frac{-1}{2\pi \mathrm{i}} \left(\mathrm{d}\alpha + (L - \iota_{X_M}\alpha)u \right), \tag{5.2}$$

is a representative of $c_1^\phi(\det(TM))$ on V. So a representative of $\beta_1 = c_1^\phi(TM)c_\phi^{n-1}$ on V is

$$\frac{-1}{2\pi \mathrm{i}} \left(\mathrm{d}\alpha + (L - \iota_{X_M}\alpha)u \right) \wedge (\omega + fu)^{n-1}. \tag{5.3}$$

On the other hand, if $G = U(1)$ and $\mu : M \to \mathfrak{u}(1)^*$ is the normalized moment map, $\mathcal{R}_\phi(Z)$ defined in (1.6) is equal to

$$\mathcal{R}_\phi(Z) = \left(\frac{1}{2\pi \mathrm{i}} \right)^n \int_M \mathrm{e}^{\mathrm{i} c_\phi(Z)}, \tag{5.4}$$

for any $Z \in \mathfrak{g}$. So Theorem 10 is applicable to $\mathcal{R}$.

5.1. Flag manifolds

Let $\eta \in \mathfrak{g}^*$ be a regular element; that is, the stabilizer G_η of η for the coadjoint action of G is a maximal torus T. By $\mathcal{O}$ is denoted the coadjoint orbit of η, endowed with the Kirillov symplectic structure ω. The G-action on $\mathcal{O}$ is Hamiltonian and the inclusion map $\mu : \mathcal{O} \to \mathfrak{g}^*$ is a moment map for this action. The Fourier transform of the orbit $\mathcal{O}$ is the function F defined on $\mathfrak{g}$ by (see [5])

$$F(X) = \left(\frac{1}{2\pi \mathrm{i}} \right)^n \int_{\mathcal{O}} \mathrm{e}^{\mathrm{i}(\mu(X)+\omega)}, \tag{5.5}$$

where $X \in \mathfrak{g}$ and $n = (\dim \mathcal{O})/2$.

Let Y be a vector of $\mathfrak{g}$, by ϕ_t^Y we denote the isotopy defined in (1.10). If $\{\phi_t^Y\}_{t\in[0,1]}$ is a closed curve in $\mathrm{Ham}(\mathcal{O})$, we have a Hamiltonian circle action $\phi^Y : U(1) \to \mathrm{Ham}(\mathcal{O})$ and $\mu(Y)$ is a Hamiltonian function for this S^1-action. If

$$\kappa := \left(\int_{\mathcal{O}} \mu(Y)\omega^n \right) \left(\int_{\mathcal{O}} \omega^n \right)^{-1}, \tag{5.6}$$

then $f = \mu(Y) - \kappa$ is the normalized Hamiltonian which generates the $U(1)$-action.

On the other hand, one deduces from (5.5)

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(tY) = \left(\frac{1}{2\pi}\right)^n \frac{\mathrm{i}}{n!} \int_{\mathcal{O}} \mu(Y)\omega^n. \tag{5.7}$$

It follows from (5.6) and (5.7) the following formula for the constant κ

$$\kappa = \frac{-\mathrm{i}}{\mathrm{Vol}(\mathcal{O})} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(tY), \tag{5.8}$$

where the symplectic volume is

$$\mathrm{Vol}(\mathcal{O}) = \frac{1}{(2\pi)^n} \frac{1}{n!} \int_{\mathcal{O}} \omega^n.$$

According to (5.4) and (5.8) we have the following proposition.

Proposition 11. *Given $Y \in \mathfrak{g}$, if ϕ^Y is a loop in* Ham($\mathcal{O}$)*, then*

$$\mathcal{R}_{\phi^Y}(Y) = \exp\left(-\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log F(tY)\right) F(Y). \tag{5.9}$$

Let W be the Weyl group determined by the torus T and X an regular element of $\mathfrak{t}$. The Harish-Chandra theorem gives a formula for $F(X)$ in terms of roots of $\mathfrak{t}$ (see [5])

$$F(X) = \prod_{\eta(i\check{\alpha})>0} (\alpha(X))^{-1} \sum_{w\in W} \epsilon(w)\mathrm{e}^{\mathrm{i}(w\cdot\eta)(X)}, \tag{5.10}$$

where $\epsilon(w)$ is the signature of the permutation $w \in W$. From (5.9) and (5.10) one deduces that $\mathcal{R}_{\phi^Y}(Y)$ can be calculated using the root structure defined by the pair (G, T).

Example. Let $G = SU(2)$ and $\eta \in \mathfrak{su}(2)^*$ defined by

$$\eta : \begin{pmatrix} b\mathrm{i} & z \\ -\bar{z} & -b\mathrm{i} \end{pmatrix} \in \mathfrak{su}(2) \mapsto b \in \mathbb{R}.$$

The stabilizer of η is $T = U(1)$, the coadjoint orbit $\mathcal{O}$ is $\mathbb{C}P^1$ and the corresponding symplectic form is ω_{area}. The Weyl group $W = N(T)/T$, with $N(T)$ the normalizer of T in $SU(2)$, consists of the class of id and the class of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.11}$$

If $\alpha_1 = \epsilon_1 - \epsilon_2$ is the usual base of roots, then $\eta(\mathrm{i}\check{\alpha}_1) = 1$. In this case the product in (5.10) has only one factor and the sum of two addends.

Let $Y = \mathrm{diag}(\pi\mathrm{i}, -\pi\mathrm{i})$, then the vector C_t in (2.3) for $n = k = 1$ is equal to tY. Thus ϕ^Y is the loop in Ham($\mathcal{O}$) denoted by ${}_1\psi$ in Section 2. This loop defines the only nontrivial class of $\pi_1(\mathrm{Ham}(\mathcal{O}))$ (see the paragraph before Theorem 5).

If w is the element of W defined by (5.11), then $wY = \mathrm{diag}(-\pi\mathrm{i}, \pi\mathrm{i})$, and $\eta(wY) = -\pi$. Furthermore $\alpha_1(Y) = 2\pi\mathrm{i}$. It follows from (5.10)

$$F(Y) = \frac{1}{2\pi\mathrm{i}}\left(\mathrm{e}^{\pi\mathrm{i}} - \mathrm{e}^{-\pi\mathrm{i}}\right) = 0.$$

By (5.9) $\mathcal{R}_{\phi^Y}(Y) = 0$.

In general, if $b \neq 0$ then for $Z = bY$,

$$F(Z) = \frac{\sin b\pi}{b\pi}.$$

The loop determined by $2Y$ defines the trivial class in $\pi_1(\mathrm{Ham}(\mathcal{O}))$, and $\mathcal{R}_{\phi^{2Y}}(2Y) = 0$.

On the other hand,

$$\mathcal{R}_{\phi^0}(X) = \mathrm{Vol}(\mathcal{O}), \tag{5.12}$$

for any $0 \neq X \in \mathfrak{g}$. It follows from Theorem 10 and (5.12) the following proposition.

Proposition 12. *The circle action on* $\left(\mathbb{C}P^1, \omega_{\text{area}}\right)$ *defined by* $\mathrm{diag}(2\pi \mathrm{i}, -2\pi \mathrm{i}) \in \mathfrak{su}(2)$ *and the trivial one determine distinct elements in* $[U(1), \mathrm{Ham}(\mathbb{C}P^1)]_{gh}$.

Proposition 12 gives an example of a pair of Hamiltonian circle actions on $(\mathbb{C}P^1, \omega_{\text{area}})$ which define the same element in $\pi_1(\mathrm{Ham})$ but they are not homotopic by a family of circle actions.

5.2. Hirzebruch surfaces

Next we determine the value $R_\phi := R_1(\phi)$ for three Hamiltonian actions on a Hirzebruch surface. We will define these actions using the fact that such a surface is a submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$.

Given 3 numbers k, τ, σ, with $k \in \mathbb{Z}_{>0}$, $\tau, \sigma \in \mathbb{R}_{>0}$ and $k\sigma < \tau$, the triple (k, τ, σ) determine a Hirzebruch surface M (see [4]). This surface is the quotient

$$\{z \in \mathbb{C}^4 : k|z_1|^2 + |z_2|^2 + |z_4|^2 = \tau/\pi,\ |z_1|^2 + |z_3|^2 = \sigma/\pi\}/\mathbb{T},$$

where the equivalence defined by $\mathbb{T} = (S^1)^2$ is given by

$$(a, b) \cdot (z_1, z_2, z_3, z_4) = (a^k b z_1, a z_2, b z_3, a z_4),$$

for $(a, b) \in (S^1)^2$.

The map

$$[z_1, z_2, z_3, z_4] \mapsto ([z_2 : z_4], [z_2^k z_3 : z_4^k z_3 : z_1])$$

allows us to represent M as a submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$. On the other hand, the usual symplectic structures on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ induce a symplectic form ω on M, and the following $(S^1)^2$-action on $\mathbb{C}P^1 \times \mathbb{C}P^2$

$$(a, b)([u_0 : u_1], [x_0 : x_1 : x_2]) = ([a u_0 : u_1], [a^k x_0 : x_1 : b x_2])$$

gives rise to a toric structure on M. The Delzant polytope associated to (M, ω) is the trapezoid in $(\mathbb{R}^2)^*$ whose not oblique edges have the lengths τ, σ, and $\lambda := \tau - k\sigma$ (see [10]). Moreover λ is the value that the symplectic form ω takes on $\{[z] \in M : z_3 = 0\}$, the exceptional divisor of M, when $k = 1$. And ω takes the value σ on the class of the fibre in the fibration $M \to \mathbb{C}P^1$.

Let ϕ_t be the diffeomorphism of M defined by

$$\phi_t[z_1, z_2, z_3, z_4] = [z_1 \mathrm{e}^{2\pi \mathrm{i} t}, z_2, z_3, z_4]. \tag{5.13}$$

$\phi = \{\phi_t : t \in [0, 1]\}$ is a loop of Hamiltonian symplectomorphisms of (M, ω). The fixed point set is $Z = \{[z] \in M : z_1 = 0\}$; that is, $Z \simeq \mathbb{C}P^1$ is the section at infinity of $M \to \mathbb{C}P^1$ (see [4]).

On M we can consider the covering

$$U_1 = \{[z] \in M : z_3 \neq 0 \neq z_4\}, \qquad U_2 = \{[z] \in M : z_1 \neq 0 \neq z_4\}$$
$$U_3 = \{[z] \in M : z_1 \neq 0 \neq z_2\}, \qquad U_4 = \{[z] \in M : z_2 \neq 0 \neq z_3\}.$$

So $Z \cap U_j = \emptyset$, for $j = 2, 3$. On U_4 one defines the complex coordinates

$$w_0 := \frac{z_4}{z_2}, \qquad w_0' := \frac{z_1}{z_3 z_2^k}.$$

In these coordinates

$$\phi_t(w_0, w_0') = (w_0, w_0' \mathrm{e}^{2\pi \mathrm{i} t}). \tag{5.14}$$

On U_1 we introduce the complex coordinates

$$w_1 := \frac{z_2}{z_4}, \qquad w_1' := \frac{z_1}{z_3 z_4^k}. \tag{5.15}$$

Thus on $U_1 \cap U_4$ one has the following relation

$$\frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_1'} = -w_0^{k+2} \frac{\partial}{\partial w_0} \wedge \frac{\partial}{\partial w_0'} \tag{5.16}$$

between the sections of det(*TM*).

On $Z \cap U_4$ we have the complex coordinate w_0 and on $Z \cap U_1$ the coordinate w_1, with $w_1 = w_0^{-1}$. By (5.16) the bundle $\det(TM)|_Z$ is the one whose first Chern class is $(k+2)$; that is, $\det(TM)|_Z = \mathcal{O}(k+2)$.

Let us consider the local section

$$s := \frac{\partial}{\partial w_0} \wedge \frac{\partial}{\partial w_0'}$$

of det(*TM*). We need to determine the corresponding function L which appears in (5.2). From (5.14) it follows

$$(\phi_t)_* \left(\frac{\partial}{\partial w_0} \right) = \frac{\partial}{\partial w_0}, \qquad (\phi_t)_* \left(\frac{\partial}{\partial w_0'} \right) = \mathrm{e}^{2\pi \mathrm{i} t} \frac{\partial}{\partial w_0'}, \tag{5.17}$$

then

$$(\phi_t)_*(s) = \mathrm{e}^{2\pi \mathrm{i} t} s.$$

Thus the above function L is the constant $2\pi\mathrm{i}$.

On the other hand, the Hamiltonian vector field X_M which corresponds to $X = 2\pi\mathrm{i} \in \mathfrak{u}(1)$ is

$$X_M = -2\pi\mathrm{i} w_0' \frac{\partial}{\partial w_0'}.$$

On $Z' := Z \setminus (\{w_0 = 0\} \cup \{w_1 = 0\})$ the class $\beta = c_1^{\phi}(TM) c_{\phi}$ is represented by the equivariant form

$$\left(\delta|_{Z'} - \frac{1}{2\pi\mathrm{i}} (2\pi\mathrm{i} - 0) u \right) (\omega|_{Z'} + f|_{Z'} u), \tag{5.18}$$

where δ is a 2-form representing the ordinary first Chern class of det(*TM*).

The normalized Hamiltonian function is $f = \pi |z_1|^2 - \kappa$, with $\kappa \in \mathbb{R}$. So $f_{|Z} = -\kappa$. The constant κ is fixed by the normalization condition. An easy calculation gives

$$\kappa = \frac{\sigma}{3} \left(\frac{3\lambda + k\sigma}{2\lambda + k\sigma} \right). \tag{5.19}$$

Next we calculate the equivariant Euler class e_Z of the normal bundle N_Z to Z in M. Let q be a point of Z', then

$$T_q M = \mathbb{C} \frac{\partial}{\partial w_0'} \oplus T_q Z.$$

Since $\frac{\partial}{\partial w_0'}$ and $\frac{\partial}{\partial w_1'}$ are sections of N_Z on $Z \cap U_4$ and $Z \cap U_1$ respectively, such that

$$\frac{\partial}{\partial w_1'} = w_0^k \frac{\partial}{\partial w_0'},$$

we have $N_Z = \mathcal{O}(k)$.

We put $w_0' = x + \mathrm{i}y$ and on N_Z we consider the orientation defined by $\frac{\partial}{\partial w_0'}$. The vector field X_M at $q = (x, y)$ is $X_M = 2\pi\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$. So

$$\left[X_M, \frac{\partial}{\partial x}\right] = 2\pi\frac{\partial}{\partial y}, \qquad \left[X_M, \frac{\partial}{\partial y}\right] = -2\pi\frac{\partial}{\partial x}.$$

Then $\det^{1/2}([X_M,\ ]) = 2\pi$. If $c_1(\mathcal{O}(k))$ is represented in Z' by the 2-form χ, then the equivariant Euler class e_Z is represented by (see [5])

$$\chi + (-2\pi)\det{}^{-1/2}([X_M,\ ])u = \chi - u.$$

It follows from (5.1) and (5.18) that

$$R_\phi = \int_{Z'}\frac{(\delta - u)(\omega - \kappa u)}{\chi - u} = \frac{-1}{u}\int_{Z'}(\delta - u)(\omega - \kappa u)(1 + \chi/u) = 2\kappa + \omega(Z). \tag{5.20}$$

A straightforward calculation gives $\int_Z \omega = \lambda + k\sigma$. It follows from this value together with (5.19) and (5.20)

$$R_\phi = \frac{6\lambda^2 + (9k+6)\lambda\sigma + (2k + 3k^2)\sigma^2}{6\lambda + 3k\sigma}. \tag{5.21}$$

Given $0 \neq r \in \mathbb{Z}$ we can consider the loop ξ defined by

$$\xi_t[z] = [z_1 e^{2\pi r t \mathrm{i}}, z_2, z_3, z_4].$$

The fixed point set for this $U(1)$-action set is Z as well. The corresponding Hamiltonian action is rf. In this case the respective function L is $2\pi r\mathrm{i}$, and now the equivariant Euler class of N_Z is $e_Z = c_1(\mathcal{O}(k)) - ru$. Hence

$$R_\xi = \frac{-1}{ru}\int_{Z'}(\delta - ru)(\omega - r\kappa u)(1 + \chi/(ru)) = R_\phi. \tag{5.22}$$

Next we shall determine $R_{\tilde\phi}$, where the $U(1)$-action $\tilde\phi_t$ is defined by

$$\tilde\phi_t[z] = [z_1, z_2 e^{2\pi t\mathrm{i}}, z_3, z_4]. \tag{5.23}$$

Now the fixed point set is $\tilde Z = \{[z] \in M : z_2 = 0\}$, it is the fibre over $[0:1]$ of the fibration $M \to \mathbb{C}P^1$ and can be identified with

$$\mathbb{C}P^1 \simeq \{([0:1], [0:z_3:z_1])\} \subset \mathbb{C}P^1 \times \mathbb{C}P^2.$$

The normalized Hamiltonian function is $\tilde f = \pi|z_2|^2 - \tilde\kappa$, with

$$\tilde\kappa = \frac{3\lambda^2 + 3k\lambda\sigma + k^2\sigma^2}{6\lambda + 3k\sigma}. \tag{5.24}$$

A calculation similar to the preceding one shows that

$$R_{\tilde\psi} = 2\tilde\kappa + \omega(\tilde Z). \tag{5.25}$$

Since $\int_{\tilde Z}\omega = \sigma$, it follows from (5.25) together with (5.24) that

$$R_{\tilde\psi} = \frac{6\lambda^2 + (6k+6)\lambda\sigma + (3k + 2k^2)\sigma^2}{6\lambda + 3k\sigma}. \tag{5.26}$$

One can consider the Hamiltonian loop $\hat\phi_t$ defined by

$$\hat\phi_t[z] = [z_1, z_2, z_3 e^{2\pi \mathrm{i} t}, z_4]. \tag{5.27}$$

The corresponding fixed point set is $\hat{Z} = \{[z] : z_3 = 0\}$. The normalized Hamiltonian function $\hat{f}$ is $\hat{f} = \pi|z_3|^2 - \hat{\kappa}$, with

$$\hat{\kappa} = \frac{3\lambda\sigma + 2k\sigma^2}{6\lambda + 3k\sigma}. \tag{5.28}$$

It is easy to prove that

$$R_{\hat{\phi}} = 2\hat{\kappa} + \omega(\hat{Z}), \tag{5.29}$$

and $\omega(\hat{Z}) = \lambda$.

We can state the following theorem.

Theorem 13. *If ϕ_t, $\tilde{\phi}_t$ and $\hat{\phi}_t$ are the loops in* $\mathrm{Ham}(M)$ *defined by (5.13), (5.23) and (5.27) respectively, then*

$$R_\phi = 2\kappa + \omega(Z), \qquad R_{\tilde{\phi}} = 2\tilde{\kappa} + \omega(\tilde{Z}), \qquad R_{\hat{\phi}} = 2\hat{\kappa} + \omega(\hat{Z}),$$

where Z, $\tilde{Z}$, and $\hat{Z}$ are the respective fixed point sets and the constants κ, $\tilde{\kappa}$, and $\hat{\kappa}$ are given by (5.19), (5.24) and (5.28) respectively.

From (5.21) and (5.26) it follows

$$R_\phi - R_{\tilde{\phi}} = \frac{3k\lambda\sigma + k(k-1)\sigma^2}{6\lambda + 3k\sigma} > 0. \tag{5.30}$$

For $0 \neq r \in \mathbb{Z}$ we denote by ϕ_t^r the diffeomorphism of M composition

$$\overbrace{\phi_t \circ \cdots \circ \phi_t}^{r}$$

if $r > 0$, and the obvious composition when $r < 0$. By (5.22) one has $R_{\phi^r} = R_\phi$. From Theorem 10 together with (5.30) one deduces the following corollary.

Corollary 14. *If r, r' are nonzero integers, then ϕ_t^r and $\tilde{\phi}_t^{r'}$ are loops in* $\mathrm{Ham}(M)$ *which are not homotopic by a homotopy consisting of Hamiltonian circle actions.*

Finally we consider the 1-parameter subgroup ζ in $\mathrm{Ham}(M)$ defined by the toric structure and the inclusion

$$y \in S^1 \mapsto (y^l, y^{\tilde{l}}) \in (S^1)^2,$$

where $l, \tilde{l} \in \mathbb{Z} \setminus \{0\}$. That is,

$$\zeta_t[z] = [z_1 e^{2\pi i l t}, z_2 e^{2\pi i \tilde{l} t}, z_3, z_4]. \tag{5.31}$$

The fixed point set F of ζ is the singleton set $F = \{[z] \in M : z_1 = z_2 = 0\}$. This point belongs to U_1; and in the coordinates w_1, w_1' (see (5.15)) on U_1

$$\zeta_t(w_1, w_1') = (w_1 e^{2\pi i l t}, w_1' e^{2\pi i \tilde{l} t}).$$

Hence

$$\zeta_{t*}\left(\frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_1'}\right) = e^{2\pi i (l+\tilde{l})t} \frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_1'},$$

and the corresponding function L is the constant $2\pi i(l + \tilde{l})$.

On the other hand, the corresponding Hamiltonian vector field Y_M is

$$Y_M = -2\pi i \left(l w_1 \frac{\partial}{\partial w_1} + \tilde{l} w_1' \frac{\partial}{\partial w_1'} \right),$$

which vanishes on F. The normalized Hamiltonian function defined by ζ is

$$l(\pi|z_1|^2 - \kappa) + \tilde{l}(\pi|z_2|^2 - \tilde{\kappa}).$$

On F this Hamiltonian reduces to the constant $-(l\kappa + \tilde{l}\tilde{\kappa})$.

According to our conventions the $U(1)$-action on $\mathbb{C}\frac{\partial}{\partial w_1}$ has multiplicity $-l$, and the multiplicity on the space $\mathbb{C}\frac{\partial}{\partial w_1'}$ is $-\tilde{l}$, then the equivariant Euler class of the normal bundle to F in M is $e_F = l\tilde{l}u^2$. Thus by (5.1) and (5.3), $R_\zeta = (l + \tilde{l})(l\kappa + \tilde{l}\tilde{\kappa})(l\tilde{l})^{-1}$.

One can state the following proposition.

Proposition 15. *Let $l, \tilde{l}$ be nonzero integers, and ζ the 1-subgroup of* Ham(M) *defined by (5.31), then*

$$R_\zeta = \frac{(l + \tilde{l})(l\kappa + \tilde{l}\tilde{\kappa})}{l\tilde{l}},$$

where the constants κ and $\tilde{\kappa}$ are given by (5.19) and by (5.24) respectively.

R_ζ is a rational function in the variables $l, \tilde{l}$. Hence $R_\zeta = R_{\zeta^r}$, for any $r \in \mathbb{Z} \setminus \{0\}$. If $\mathbf{l} = (l, \tilde{l})$ and $\mathbf{l}' = (l', \tilde{l}')$ are two pairs of nonzero integers such that the corresponding 1-parameter subgroups $\zeta(\mathbf{l})$ and $\zeta(\mathbf{l}')$ satisfy $R_{\zeta(\mathbf{l})} \neq R_{\zeta(\mathbf{l}')}$, then, by Theorem 10, $\zeta(\mathbf{l})^r$ and $\zeta(\mathbf{l}')^s$ are not homotopic by a family of Hamiltonian S^1-actions, whenever $r, s \in \mathbb{Z}\setminus\{0\}$.

When M is the Hirzebruch surface determined by the triple $(k = 1, \tau > 2, \sigma = 1)$, Abreu and McDuff proved in [2] that $\pi_1(\mathrm{Ham}(M))$ is isomorphic to $\mathbb{Z}$. Thus we have the following corollary.

Corollary 16. *Let M be the Hirzebruch surface defined by* $(k = 1, \tau > 2, \sigma = 1)$. *There are infinitely many pairs* $(\zeta(\mathbf{l}), \zeta(\mathbf{l}'))$ *of 1-parameter closed subgroups of* Ham(M) *such that* $[\zeta(\mathbf{l})] = [\zeta(\mathbf{l}')] \in \pi_1(\mathrm{Ham}(M))$, *but*

$$[\zeta(\mathbf{l})] \neq [\zeta(\mathbf{l}')] \in [U(1), \mathrm{Ham}(M)]_{gh}.$$

Remark 1. Two Hamiltonian circle actions on M, ϕ and ϕ' are conjugate if there exists an element $h \in \mathrm{Ham}(M)$, such that $h \cdot \phi_t \cdot h^{-1} = \phi_t'$ for all t. If ϕ and ϕ' are conjugate, let h_s be a path in Ham(M) from Id to h, then $h_s \cdot \phi \cdot h_s^{-1}$ defines a homotopy between ϕ and ϕ', and $[\phi] = [\phi'] \in [U(1), \mathrm{Ham}(M)]_{gh}$. By Corollary 16, there are infinitely many conjugacy classes of circle actions on the Hirzebruch surface considered in this corollary.

Remark 2. Although the characteristic R_1 allows us to distinguish infinitely many conjugacy classes of Hamiltonian circle actions in a Hirzebruch surface M, the situation is different for $U(1)^2$-actions, as we show next.

Given $\mathbf{l} = (l, \tilde{l})$ a pair of nonzero integers, we define a $U(1)^2$-action on M by

$$\xi_{st}[z] = [z_1 e^{2\pi i l s}, z_2 e^{2\pi i \tilde{l} t}, z_3, z_4].$$

The fixed point set is again the singleton F, and the equivariant Euler class $e_F = l\tilde{l}uv$, where u, v are coordinates on $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$. According to the localization formula (5.1), $R_1(\xi)$ is a rational function of the variables $\check{u} := lu$, $\check{v} := \tilde{l}v$. The numerator is a homogeneous degree two polynomial $A_1\check{u}^2 + A_2\check{u}\check{v} + A_3\check{v}^2$, with A_j independent of $\mathbf{l}$; and the denominator is $\check{u}\check{v}$. As $R_1(\xi) \in H^0(B(U(1)^2))$, then $A_1 = A_3 = 0$, and $R_1(\xi)$ is independent of $\mathbf{l}$. That is, R_1 is constant on $\{\xi(\mathbf{l})\}_{\mathbf{l}}$. But Karshon proved that the number of conjugacy classes of maximal tori in M is the smallest integer greater than or equal to $\frac{\lambda}{\sigma}$ (see [15]). That is, the characteristic number R_1 is not fine enough to analyze this case.

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